Particle on a sphere

$$-\frac{\hbar^2}{2I}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]Y_{l m_l}(\theta,\phi)=E\,Y_{l m_l}(\theta,\phi)\quad\ldots\ (1)$$

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]Y_{l m_l}(\theta,\phi)=-\frac{2IE}{\hbar^2}Y_{l m_l}(\theta,\phi)\quad\ldots\ (2)$$

$$\beta=\frac{2IE}{\hbar^2}$$

also $\beta = l(l+1)$ $\rightarrow$ $E=\frac{\hbar^2}{2I}l(l+1)$

$l = 0, 1, 2, 3 \ldots$

$$-\frac{\hbar^2}{2I}\Lambda^2 Y_{l m_l}(\theta,\phi)=\frac{\hbar^2}{2I}l(l+1)\,Y_{l m_l}(\theta,\phi)\quad\ldots\ (3)$$

$$K.E. = T = \frac{1}{2}I\omega^2 \ ; \quad L = I\omega$$

$$T = \frac{L^2}{2I}$$

for particle in a 1-D box $\quad T=\frac{p_x^2}{2m}$

$$\hat{T}=\frac{\hat{p}_x^2}{2m}$$

$$\frac{\hat{p}_x^2}{2m}\psi(x)=E\,\psi(x)$$

$$\hat{T} = \frac{\hat{L}^2}{2I}$$

$$\frac{\hat{L}^2}{2I} Y_{lm_l}(\theta, \phi) = E\, Y_{lm_l}(\theta, \phi) \quad \cdots (4)$$

compare (1) and (4)

$$\hat{L}^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right] \quad \cdots (5)$$

from (4)

$$\frac{\hat{L}^2}{2I} Y_{lm_l}(\theta, \phi) = \frac{\hbar^2}{2I} l(l+1) Y_{lm_l}(\theta, \phi) \quad \cdots (6)$$

$$\hat{L}^2 Y_{lm_l}(\theta, \phi) = \hbar^2 l(l+1) Y_{lm_l}(\theta, \phi) \quad \cdots (7)$$

$$|L| = \sqrt{\hbar^2 l(l+1)} = \hbar \sqrt{l(l+1)}$$

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 \quad \cdots (8)$$

$$\hat{L}_z = -i\hbar \frac{\partial}{\partial\phi}$$

$$\hat{L}_z Y_{lm_l}(\theta, \phi) = m_l \hbar Y_{lm_l}(\theta, \phi) \quad \cdots (9)$$

$$\hat{L}_z^2 Y_{lm_l}(\theta, \phi) = m_l^2 \hbar^2 Y_{lm_l}(\theta, \phi) \quad \cdots (10)$$

from eqn. (8) $\quad \hat{L}^2 - \hat{L}_z^2 = \hat{L}_x^2 + \hat{L}_y^2 \quad \cdots (11)$

$$\left[\hat{L}^2 - \hat{L}_z^2\right] Y_{lm_l}(\theta, \phi) = \left[\hbar^2 l(l+1) - m_l^2 \hbar^2\right] Y_{lm_l}(\theta, \phi)$$

$$\cancel{(\hat{L}_x^2 + \hat{L}_y^2)} =$$

$$\left[\hat{L}_x^2 + \hat{L}_y^2\right] Y_{lm_l}(\theta, \phi) = \hbar^2 \left[l(l+1) - m_l^2\right] Y_{lm_l}(\theta, \phi) \quad \ldots (2)$$

Space Quantitation

$$\cos\theta = \frac{m_l \hbar}{\hbar \sqrt{l(l+1)}}$$

$$\cos\theta = \frac{m_l}{\sqrt{l(l+1)}} \quad \Big\} \text{ angular orientations possible!}$$

Commutation Relations

$$\begin{tikzcd} & \hat{L}_x \arrow[dr, bend left] & \\ \hat{L}_z \arrow[ur, bend left] & & \hat{L}_y \arrow[ll, bend left] \end{tikzcd}$$

$$[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z$$

$$[\hat{L}_y, \hat{L}_z] = i\hbar \hat{L}_x$$

$$[\hat{L}_z, \hat{L}_x] = i\hbar \hat{L}_y$$

$$[\hat{L}^2, \hat{L}_z] = [\hat{L}^2, \hat{L}_y] = [\hat{L}^2, \hat{L}_x] = 0$$

$$\Phi(\phi) = \frac{1}{\sqrt{2\pi}} e^{i m_l \phi} \qquad m_l = 0, \pm 1, \pm 2 \cdots$$

$$\int_0^{2\pi} \Phi^*_{m_l'}(\phi)\, \Phi_{m_l}(\phi)\, d\phi$$

$$= \delta_{m_l' m_l}$$

$$\delta_{m_l' m_l} = 1 \quad \text{if } m_l' = m_l \quad \text{(normalisation)}$$

$$= 0 \quad \text{if } m_l' \neq m_l \quad \text{(orthogonality)}$$

$$Y_{l m_l}(\theta, \phi) = \Theta(\theta)\, \Phi(\phi)$$

$\uparrow$

spherical harmonics

$$\int_0^{2\pi} d\phi \int_0^{\pi} Y^*_{l', m_l'}(\theta, \phi)\, Y_{l, m_l}(\theta, \phi) \sin\theta\, d\theta$$

$$= \delta_{l' l}\, \delta_{m_l' m_l}$$

$$\delta_{l' l}, \delta_{m_l' m_l} = 1 \quad \text{if } l = l' \text{ and } m_l = m_l'$$

$$= 0 \quad \text{if either } l \neq l' \text{ or } m_l \neq m_l'$$

$$d\tau = r^2 dr \sin\theta \, d\theta \, d\phi$$

$$\int_0^a r^2 dr \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

$$\frac{a^3}{3} \cdot 2 \cdot 2\pi = \underbrace{\frac{4}{3}\pi a^3}_{\text{volume of sphere}}$$

$$\hat{L}_z Y_{l m_l}(\theta, \phi) = m_l \hbar Y_{l m_l}(\theta, \phi)$$

$$\hat{L}^2 Y_{l m_l}(\theta, \phi) = \hbar^2 l(l+1) Y_{l m_l}(\theta, \phi)$$

Rotational Motion → Spectroscopy

$$E = \frac{\hbar^2}{2I}(J+1)J$$